→ This is nothing but the rotation around $\hat{z}$-axis
with angle $\phi = \omega t$!

But, there's a weird thing.

- $\langle \vec{S} \rangle_{2\pi} = \langle \vec{S} \rangle_0$; It's ok.

$$|\alpha, t\rangle = \mathcal{U}(t) \left[|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow| \right] |\alpha\rangle \qquad \| \ \phi = \omega t.$$

$$= e^{-\frac{i\phi}{2}} |\uparrow\rangle\langle\uparrow|\alpha\rangle + e^{\frac{i\phi}{2}} |\downarrow\rangle\langle\downarrow|\alpha\rangle$$

$$|\alpha, 2\pi\rangle = - |\alpha, 0\rangle. \quad !!!$$

The state comes back with a minus sign!

or.
↳ pressesion period $\tau = \frac{2\pi}{\omega}$ for $\langle \vec{S} \rangle$.

but $\tau_{\text{state ket}} = \text{"}\frac{4\pi}{\omega}\text{"}$ for $|\alpha\rangle$.

(3) generalization : SU(2) vs. SO(3)

- Pauli two-component formalism
with the "Pauli" spinor.

ket

$$|\uparrow\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv \chi_\uparrow \ , \quad |\downarrow\rangle \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv \chi_\downarrow$$

bra

$$\langle \downarrow| \doteq (1, 0) \equiv \chi_\uparrow^\dagger \ , \quad \langle\downarrow| \doteq (0, 1) \equiv \chi_\downarrow^\dagger$$

a state

$$|\alpha\rangle \doteq \begin{pmatrix} \langle\uparrow|\alpha\rangle \\ \langle\downarrow|\alpha\rangle \end{pmatrix} , \quad \langle\alpha| \doteq (\langle\alpha|\uparrow\rangle , \langle\alpha|\downarrow\rangle).$$

$\Rightarrow$ two-component "Pauli" spinor.

$$\chi = \begin{pmatrix} \langle\uparrow|\alpha\rangle \\ \langle\downarrow|\alpha\rangle \end{pmatrix} = \begin{pmatrix} c_\uparrow \\ c_\downarrow \end{pmatrix} = c_\uparrow \chi_\uparrow + c_\downarrow \chi_\downarrow$$

and

$$\chi^\dagger = (\langle\alpha|\uparrow\rangle, \langle\alpha|\downarrow\rangle) = (c_\uparrow^*, c_\downarrow^*)$$

- Pauli Matrices

def. $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

$$\Rightarrow \tilde{S}_k \doteq \frac{\hbar}{2} \sigma_k$$

ex. $\langle S_k \rangle = \langle\alpha|S_k|\alpha\rangle = \frac{\hbar}{2} \chi^\dagger \sigma_k \chi$ ← Try to verify this.

$\Rightarrow$ properties :

$$\sigma_i^2 = 1$$
$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}$$
$$[\sigma_i, \sigma_j] = 2i\varepsilon_{ijk}\sigma_k .$$

also,

$\sigma_i^\dagger = \sigma_i$: Hermitian.

$\det[\sigma_i] = -1$: "special"

$\mathrm{Tr}[\sigma_i] = 0$: traceless.

Now, consider a vector $\vec{x} = (x, y, z) \in \mathbb{R}$ in the basis of Pauli matrices :

$$X = x\sigma_1 + y\sigma_2 + z\sigma_3$$
$$= \begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix}$$

: Hermitian, traceless.

length of the vector $|\vec{x}|^2 = x^2+y^2+z^2 = -\det X$.

$\rightarrow$ A rotation can be described by a unitary transformation,

$$X' = U X U^{-1}, \qquad \| \ \det U = 1$$

$$X' = \vec{x}' \cdot \vec{\sigma}$$

$\Rightarrow \quad |\vec{x}'|^2 = |\vec{x}|^2 \Leftarrow \det X' = \det X$. //

$\therefore$ U (a 2x2 matrix) is a rotation matrix,

mapping $SU(2)\ [U]$ onto $SO(3)\ [R]$.

special ← S; U = unitary; (2) = dimension of "defining" or "fundamental" representation, → 2x2 matrix.

: $\det U = 1$

Since U is a 2x2 matrix, it can be written as

$$U = q_0 + i\vec{\sigma}\cdot\vec{q} \qquad \| \ \vec{q} = (q_1, q_2, q_3).$$

see HW 2.1

$$UU^\dagger = 1 \Rightarrow |q_0|^2 + |\vec{q}|^2 + i\vec{\sigma}\cdot(\vec{q}\, q_0^* - c.c.) + i\vec{\sigma}\cdot(\vec{q}\times\vec{q}^*) = 1$$

* use the identity $(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b} + i\vec{\sigma}\cdot(\vec{a}\times\vec{b})$.

$\Rightarrow$ • q_0 and $\vec{q}$ are chosen to be real. (to remove $\vec{\sigma}$-dependence)

• $q_0^2 + |\vec{q}|^2 = 1$. //

Choosing $q_0 = \cos\frac{1}{2}\theta$, $\vec{q} = -\hat{z}\sin\frac{1}{2}\theta$,

$$U X U^{-1} \rightarrow \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \\ z \end{pmatrix}$$

It rotates $\vec{x}$ by θ around z-axis.

Check : $U = \cos\frac{\theta}{2} - i\sigma_3 \sin\frac{\theta}{2} = \exp\left[-i\frac{\sigma_3}{2}\theta\right] = \exp\left[-\frac{i S_z}{\hbar}\theta\right]$.

a general rotation by angle ϕ around $\hat{n}$-axis.

: $q_0 = \cos\frac{1}{2}\phi$, $\vec{q} = -\hat{n}\sin\frac{1}{2}\phi$

$\rightarrow$ $U = \exp\left[-\frac{i}{2}(\vec{\sigma}\cdot\hat{n})\phi\right]$. || This is the case where $\vec{J} = \frac{\hbar}{2}\vec{\sigma}$

verification

$$U = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!}\left(\frac{\phi}{2}\right)^n (\hat{n}\cdot\vec{\sigma})^n$$

by using $(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b} + i\vec{\sigma}\cdot(\vec{a}\times\vec{b})$

$\Rightarrow (\hat{n}\cdot\vec{\sigma})^{2n} = 1$

$$= \left[\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\left(\frac{\phi}{2}\right)^{2k}\right]\cdot I - i(\hat{n}\cdot\vec{\sigma})\left[\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\left(\frac{\phi}{2}\right)^{2k+1}\right]$$

$$= \cos\frac{\phi}{2}\cdot I - i(\hat{n}\cdot\vec{\sigma})\sin\frac{\phi}{2}.$$

* U has the period of 4π ! Does it sound reasonable?

Yes. $SU(2)$ covers $SO(3)$ twice !

"Cayley-Klein" parameters

In another general form, $U(a,b) = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$

with $|a|^2 + |b|^2 = 1$.

$\Rightarrow$ $U^\dagger(a,b)\, X\, U(a,b) = X'$ || $U(-a,-b) = -U(a,b)$

$U^\dagger(-a,-b)\, X\, U(-a,-b) = X'$

U and $-U$ generates the same R.

$[2\pi] + [2\pi] \longrightarrow [2\pi]$.

$\therefore$ A state ket rotated by U has 4π-periodicity !

$= U(\hat{n},\phi)|\alpha\rangle$ in $SU(2)$.

(4) Eigenvalues and Eigenstates of $\vec{J}$ and J^2.

- Commutation Relations and Ladder Operators

Lie Algebra : $[J_i, J_j] = i\hbar \varepsilon_{ijk} J_k$

Everything starts from this relation.

Casimir operator

$\Rightarrow$ $[J^2, J_k] = 0$. $\quad \| \quad J^2 = J_x^2 + J_y^2 + J_z^2$

$\therefore$ There are simultaneous eigenkets of J^2 and J_k.

$$\Rightarrow \quad J^2|a,b\rangle = a|a,b\rangle$$
$$J_z|a,b\rangle = b|a,b\rangle$$

|| NOTE: There's a typo in S&N, 2nd ed..

def. Ladder operators ; Let's see how these work.

$$J_\pm \equiv J_x \pm iJ_y$$

Commutation relations : $[J_+, J_-] = 2\hbar J_z$

$[J_z, J_\pm] = \pm\hbar J_\pm$

$[J^2, J_\pm] = 0.$

why "ladder"?

$$\Rightarrow \quad J_z(J_\pm|a,b\rangle) = ([J_z, J_\pm] + J_\pm J_z)|a,b\rangle$$
$$= (b \pm \hbar)(J_\pm|a,b\rangle)$$

$\therefore$ It raises or lowers the eigenvalue b.

But it doesn't change "a" since $[J^2, J_\pm] = 0$.

$$\Rightarrow \quad J^2(J_\pm|a,b\rangle) = a(J_\pm|a,b\rangle).$$

c-number.

Therefore, we may write it as $\boxed{J_\pm|a,b\rangle = C_\pm|a, b\pm\hbar\rangle}$.

- Eigen values of J^2 and J_z.

Can we apply $J_\pm$ again and again, indefinitely? NO.

Consider $$J^2 - J_z^2 = \frac{1}{2}(J_+J_- + J_-J_+)$$
$$= \frac{1}{2}(J_-^\dagger J_- + J_+^\dagger J_+)$$

$$\rightarrow \langle a,b|J^2 - J_z^2|a,b\rangle = \frac{1}{2}[\langle -|-\rangle + \langle +|+\rangle]$$
$$\geq 0 \qquad \Big\| \quad |-\rangle = J_-|a,b\rangle, \quad |+\rangle = J_+|a,b\rangle$$

$$\Rightarrow \quad a \geq b^2$$

b has upper and lower bounds given by a.

Thus, $J_+|a, b_{max}\rangle = 0 \rightarrow J_-J_+|a,b_{max}\rangle = 0$.

$$\rightarrow (J^2 - J_z^2 - \hbar J_z)|a, b_{max}\rangle = 0$$

proof. $J_-J_+ = J_x^2 + J_y^2 - \hbar J_z = J^2 - J_z^2 - \hbar J_z$.

$$\Rightarrow \quad a - b_{max}^2 - b_{max}\hbar = 0.$$

or $$a = b_{max}(b_{max} + \hbar) \quad \cdots \text{①}$$

Similarly, $J_-|a,b_{min}\rangle = 0 \rightarrow J_+J_-|a,b_{min}\rangle = 0$.

$$\rightarrow (J^2 - J_z^2 + \hbar J_z)|a,b_{min}\rangle = 0.$$

$$\Rightarrow \quad a = b_{min}(b_{min} - \hbar) \quad \cdots \text{②}$$

$$\rightarrow \text{①} - \text{②}: \quad (b_{max}^2 - b_{min}^2) + \hbar(b_{max} + b_{min}) = 0.$$

$$\rightarrow \quad \boxed{b_{min} = -b_{max}}$$

Since we can reach b_{max} by applying J_+ to $|b_{min}\rangle$ a finite number of times,

$$b_{max} = b_{min} + n\hbar .$$

n : integer, ≥ 0

$$\Downarrow$$

Define $j = \frac{n}{2} = \frac{1}{2}, 1, \frac{3}{2}, \cdots$

$$\left[\begin{array}{l} b_{max} = \frac{n\hbar}{2} \\ b_{min} = -\frac{n\hbar}{2} \end{array}\right.$$

$$\downarrow$$

let,

$$a = \hbar^2 j(j+1) \quad \text{and} \quad b \equiv m\hbar$$

The allowed $m = -j, -j+1, \cdots, j-1, j$

$$\therefore \left[\begin{array}{l} J^2 |j,m\rangle = j(j+1)\hbar^2 |j,m\rangle \\ J_z |j,m\rangle = m\hbar |j,m\rangle \end{array}\right. \qquad // \quad j = \frac{1}{2}, 1, \frac{3}{2}, \cdots$$

$\uparrow$ a half integer!

: This is a direct outcome of the Lie Algebra; We did not use anything else.

(5) Matrix elements of $\vec{J}$ and $\mathcal{D}(R)$.

- $J^2, J_z, J_\pm$

obviously, $\langle j', m' | J^2 | j, m\rangle = j(j+1)\hbar^2 \delta_{j'j} \delta_{m'm}$

$\langle j', m' | J_z | j, m\rangle = m\hbar \delta_{jj'} \delta_{mm'}$.

For J_+, we know $J_+ |j, m\rangle = C^{(+)}_{jm} |j, m+1\rangle$.

$$\Rightarrow \langle j,m | J_+^\dagger J_+ | j,m\rangle = \langle j,m | (J^2 - J_z^2 - \hbar J_z) | j,m\rangle$$

$$= \hbar^2 [j(j+1) - m^2 - m]$$

$$\therefore \left| C^{(+)}_{jm} \right|^2 = \hbar^2 [j(j+1) - m^2 - m] = \hbar^2 (j-m)(j+m+1)$$

Choosing $C_{jm}^{(\pm)}$ to be real and positive,

$$J_\pm |j,m\rangle = \sqrt{(j\mp m)(j\pm m+1)}\,\hbar\,|j,m\pm 1\rangle .$$

(You can check this for J_- similarly.)

$$\Rightarrow \langle j',m'|J_\pm|j,m\rangle = \sqrt{(j\mp m)(j\pm m+1)}\cdot\hbar\cdot\delta_{j'j}\,\delta_{m',m\pm 1}$$

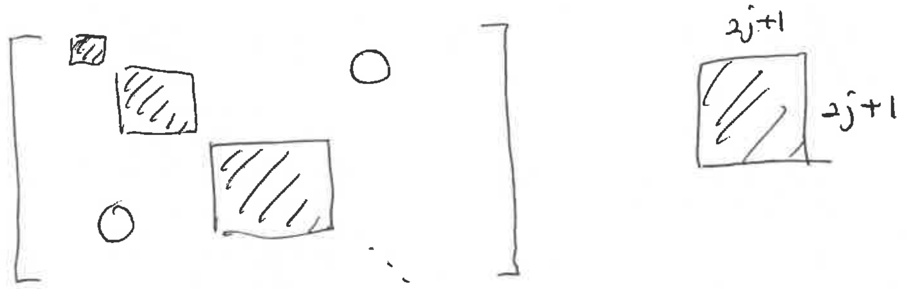
• Representations of the Rotation Operator

$$\mathcal{D}^{(j)}_{m'm}(R) \equiv \langle j,m'|\exp\left[-\frac{i}{\hbar}(\vec{J}\cdot\hat{n})\phi\right]|j,m\rangle$$

(Wigner function) or d-matrix : a matrix element of $\mathcal{D}(R)$

NOTE : it's diagonal in $|j\rangle$. $\Vert$ $\vec{J}|j\rangle \propto |j\rangle$ //

↳ a block-diagonal matrix



$(m=-j\cdots j)$

→ The rotation matrices characterized by definite j : — $(2j+1)$ by $(2j+1)$ — form a "group".

$\Vert$ NOTE : 2 is the dimension of the "defining, fundamental" one. on SU(2)

- Identity : $\phi=0$.
- Inverse : $\phi \to -\phi$
- Composition :

$$\sum_{m'} \mathcal{D}^{(j)}_{m''m'}(R_1)\,\mathcal{D}^{(j)}_{m'm}(R_2) = \mathcal{D}^{(j)}_{m''m}(R_1R_2).$$